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Foundations of Quantum Probability

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This paper presents an overview of the foundations of quantum probability. The main concepts in this theory are measurements and generalized actions. These concepts correspond to the usual quantum observables and states. Probabilities are computed by means of a universal influence function. We first derive the form of the universal influence function and then construct the amplitude and probability of a measurement with respect to a given generalized action. It is shown that traditional quantum mechanics can be derived as a special case of this theory and moreover the theory gives a complete realistic interpretation of quantum mechanics. It is demonstrated that spins of any order can be described within this framework and a realistic solution to the EPR problem can be achieved.

1. INTRODUCTION

There are various approaches to quantum probability and this overview will present one of them (Gudder, 1988*a,b*, 1989, 1992*b*). Although quantum mechanics has been with us for most of this century, we still lack a deep understanding of it. There remains something mysterious and puzzling about quantum mechanics. Much of this mystery involves its Hilbert space formulation. The traditional Hilbert space framework of quantum mechanics entails various puzzling questions, some of which are included in the following list.

1. Where does the Hilbert space H come from?
2. Why are states represented by vectors in H and observables by self-adjoint operators on H?
3. Why does the probability have its postulated form?
4. Why do the position and momentum operators have their particular form?

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5. Where do the Bohr correspondence principle and Schrödinger's equation come from?
6. Why does a physical theory which must give real-valued results involve a complex amplitude or state?
7. Why must a quantum particle exhibit wave behavior (wave–particle duality)?
8. Must quantum mechanics be nonrealistic (a quantum system only has properties when they are observed)?
9. Is there a realistic description of quantum mechanics (hidden variables model)?
10. Is there a realistic solution to the EPR problem (can quantum mechanics be completed to include all the relevant elements of reality)?

In this overview, we shall attempt to answer these questions and others that naturally arise. In doing this, we shall employ a reformulation of the mathematical foundations of quantum mechanics and the basic tenets of probability theory. This reformulation of quantum mechanics is based on the concepts of measurement, generalized action, and a unique universal influence function. The main axiom is that the probability of a measurement outcome is the sum (or integral) of the influences between pairs of alternatives that result in that outcome when the measurement is executed. Our formulation not only extends the usual quantum formalism, it contains more information about a physical system since it is based upon a deeper subquantum reality which we shall call a sample space. As in classical probability theory, the elements of the sample space represent possible alternatives or configurations of the physical system. Measurements performed on the system are represented by functions on the sample space similar to random variables. This already presents an advantage over traditional quantum mechanics since it dispenses with self-adjoint operators and replaces them with measurement functions that are easier to analyze. Although the usual self-adjoint operators can be derived if necessary, the measurements functions are more closely related to random variables and they provide a natural framework for the study of quantum stochastic processes. This formulation is also related to the Feynman formalism and gives a rigorous alternative to Feynman path integrals (Feynman, 1949; Feynman and Hibbs, 1965).

2. GENERALIZED ACTION AND UNIVERSAL INFLUENCE FUNCTION

We denote the set of possible configurations of a physical system $\mathcal{S}$ by Ω and call Ω a *sample space*. If X is a measurement on $\mathcal{S}$, then executing

X results in a unique outcome depending on the configuration ω of $\mathscr{S}$. In this way, X can be identified with a function $X: \Omega \to R(X)$, where $R(X)$ is the range of X. To be precise, we define a *measurement* to be a map $X: \Omega \to R(X)$ satisfying:

(M1) $R(X)$ is the base space of a measure space $(R(X), \Sigma_X, \mu_X)$.

(M2) For every $x \in R(X)$, $X^{-1}(x)$ is the base space of a measure space $(X^{-1}(x), \Sigma_X^x, \mu_X^x)$.

We call the elements of $R(X)$, *X-outcomes*, and the sets in Σ_X are *X-events*. Notice that $X^{-1}(x)$ corresponds to the set of configurations resulting in the outcome x when X is executed. We call the set $X^{-1}(x)$ the *X-fiber* over x. The measures $\mu_X, \mu_X^x, x \in R(X)$, represent *a priori* weights due to our knowledge of the system (for example, we may know the energy of the system or we might assume the energy has a certain value). In the case of total ignorance, these weights are taken to be counting measure in the discrete case and uniform measure in the continuous case.

The measurements correspond to the observables of traditional quantum mechanics. Notice that at this stage, we do not have a Hilbert space and we do not have self-adjoint operators representing observables. As we shall later see, these as well as the other quantum mechanical constructs can be derived from deeper fundamental principles. Moreover, this framework gives a realistic theory, since a configuration ω determines the properties of $\mathscr{S}$ independent of any particular measurement. The configurations can also be viewed as hidden variables since an $\omega \in \Omega$ completely determines the results of all measurements simultaneously. In fact, measurements are quite similar to the dynamical variables of classical mechanics and this fact will be exploited in the next section.

We next assume the existence of a real-valued function $S: D_S \to \mathbb{R}$ which we call a *generalized action* for the system $\mathscr{S}$, where $D_S \subseteq \Omega$ and $D_S \cap X^{-1}(x) \in \Sigma_X^x$ for every $x \in R(X)$. The function S depends on our model of $\mathscr{S}$ and also on our state of knowledge of $\mathscr{S}$. Moreover, we assume the existence of an *influence function* $G: \mathbb{R} \to \mathbb{R}$ and define the *influence* between $\omega, \omega' \in D_S$ *relative to* S to be

$$F_S(\omega, \omega') = N_S^2 G[S(\omega) - S(\omega')] \tag{2.1}$$

where $N_S > 0$ is a normalization constant. It will turn out that G is symmetric, so that $F_S(\omega, \omega') = F_S(\omega', \omega)$.

We can also define an influence for a superposition of two generalized actions. If S_1, S_2 are generalized actions and $a, b \in \mathbb{R}$, then the influence between $\omega, \omega' \in D_{S_1} \cap D_{S_2}$ relative to a *superposition* of S_1 and S_2 is

$$F_{S_1,S_2}(\omega, \omega') = a^2 G[S_1(\omega) - S_1(\omega')] + b^2 G[S_2(\omega) - S_2(\omega')] + 2ab G[S_1(\omega) - S_2(\omega')]$$

This can be thought of as the influence relative to S_1 plus the influence relative to S_2 plus a cross influence relative to S_1 and S_2. Notice that if $S_1 = S_2$, then this reduces to

$$F_{S_1, S_1}(\omega, \omega') = (a+b)^2 F_{S_1}(\omega, \omega')$$

which is essentially the same as F_{S_1}.

Following Hemion (1988, 1990), we now make a fundamental reformulation of the probability concept. We postulate that the probability density $P_{X,S}(x)$ of an X-outcome x is the sum (or integral) of the influences between each pair of configurations that result in x upon executing X. In precise mathematical form, we postulate that $F_S(\omega, \omega')$ is integrable and

$$P_{X,S}(x) = \int_{X^{-1}(x) \cap D_S} \int_{X^{-1}(x) \cap D_S} F_S(\omega, \omega')\, \mu_X^x(d\omega)\, \mu_X^x(d\omega') \tag{2.2}$$

Moreover, to ensure that $P_{X,S}(x)$ is indeed a probability density, we assume that $P_{X,S}$ is measurable with respect to Σ_X and the following normalization condition holds:

$$\int_{R(X)} P_{X,S}(x)\, \mu_X(dx) = 1 \tag{2.3}$$

Equation (2.3) can be used to find the normalization constant N_S.

If $B \in \Sigma_X$ is an X-event, we define the *(X, S)-probability* of B by

$$P_{X,S}(B) = \int_B P_{X,S}(x)\, \mu_X(dx) \tag{2.4}$$

We shall show later that G has a special form which implies the nonnegativity of $P_{X,S}(x)$. For this reason $P_{X,S}$ is indeed a probability measure on Σ_X, which we call the *S-distribution* of X. Similar definitions apply to superpositions of generalized actions by replacing F_S with F_{S_1, S_2} to get the probability density $P_{X,S_1,S_2}(x)$ and the distribution $P_{X,S_1,S_2}(B)$.

Influence is a strictly quantum mechanical phenomenon which is not present in classical physics. In the classical limit, $F_S(\omega, \omega')$ approaches a delta function $\delta_\omega(\omega')$. In this limit $F_S(\omega, \omega') = 0$ for $\omega \neq \omega'$ and there is no influence between distinct configurations. We then have

$$P_{X,S}(x) = \mu_X^x(X^{-1}(x))$$

and this reduces to a classical probability framework.

We can extend this theory to include expectations of functions on Ω. Let $g: \Omega \to \mathbb{R}$ be a function that is integrable along X-fibers. We define the *(X, S)-expectation of g at x* by

$$E_{X,S}(g)(x) = \int_{X^{-1}(x) \cap D_S} \int_{X^{-1}(x) \cap D_S} g(\omega) F(\omega, \omega')\, \mu_X^x(d\omega)\, \mu_X^x(d\omega')$$

This equation is the natural generalization of (2.2) from a probability to an expectation. If this last expression is integrable, then the (X, S)-*expectation* of g is given by

$$E_{X,S}(g) = \int_{R(X)} E_{X,S}(g)(x)\, \mu_X(dx) \tag{2.5}$$

We can also use this formalism to compute probabilities of events in Ω. Let $A \subset \Omega$ and denote the characteristic function of A by χ_A. If χ_A is integrable along X-fibers, in analogy with classical probability theory, we define the (X, S)-*pseudoprobability* of A by $\hat{P}_{X,S}(A) = E_{X,S}(\chi_A)$. It follows from (2.3) and (2.5) that $\hat{P}_{X,S}(\Omega) = 1$, and $\hat{P}_{X,S}$ is countably additive. However, $\hat{P}_{X,S}$ may have negative values, which is why it is called a pseudoprobability. Nevertheless, there are σ-algebras of subsets of Ω on which $\hat{P}_{X,S}$ is a probability measure. For example, if $A = X^{-1}(B)$ for $B \in \Sigma_X$, then it can be shown that $\hat{P}_{X,S}(A) = P_{X,S}(B)$ (Gudder, 1992*b*). Hence, in this case, $\hat{P}_{X,S}$ reduces to the probability distribution $P_{X,S}$. A less trivial example is given in the next section.

Until now we have not imposed any conditions on the influence function G except measurability. It turns out that, due to fundamental physical principles, G can be uniquely specified. The following conditions can be physically justified (Gudder, 1991; Hemion, 1988, 1990):

(1) G is continuous.
(2) G has a zero.
(3) G is *causal*, that is

$$\sum_{i=1}^{n} G(\theta_i) = 0 \Rightarrow \sum_{i=1}^{n} [G(\phi + \theta_i) + G(\phi - \theta_i)] = 0$$

for all $\phi \in \mathbb{R}$.

Condition (3) follows from the principle of strong causality, which states that the future cannot influence the present. The following theorem is due to Hemion (1988, 1990) and Gudder (1991).

Theorem. If $u\colon \mathbb{R} \to \mathbb{R}$ is causal, continuous, and has a zero, then there exists an $a > 0$ such that $u(\theta) = u(0) \cos a\theta$ for all $\theta \in \mathbb{R}$.

We conclude that an influence function G is essentially unique and in fact $G(\theta) = G(0) \cos a\theta$. This shows that a quantum system automatically possesses a periodic behavior and has an intrinsic wavelength. In a sense, we have derived the de Broglie wave associated with a quantum particle. By a change of scale, we can assume that $G(0) = a = 1$. We then call $G(\theta) = \cos\theta$ the *universal influence function.*

We now employ the universal influence function G in our previous probability formulas. Equation (2.1) now becomes

$$F_S(\omega, \omega') = N_S^2 \cos[S(\omega) - S(\omega')] \tag{2.6}$$

Substituting (2.6) into (2.2) gives

$$P_{X,S}(x) = \left| \int_{X^{-1}(x) \cap D_S} N_S e^{iS(\omega)} \, \mu_X^x(d\omega) \right|^2 \tag{2.7}$$

We call the function

$$f_S(\omega) = \begin{cases} N_S e^{iS(\omega)} & \text{for} \quad \omega \in D_S \\ 0 & \text{for} \quad \omega \notin D_S \end{cases}$$

the S-*amplitude function* and define the (X, S)-*wave function* by

$$f_{X,S}(x) = \int_{X^{-1}(x)} f_S(\omega) \, \mu_X^x(d\omega) \tag{2.8}$$

From (2.7) and (2.8) we have

$$P_{X,S}(x) = |f_{X,S}(x)|^2 \tag{2.9}$$

Equation (2.7) shows how the complex numbers arise in quantum mechanics. The complex numbers are not needed for the computation of $P_{X,S}(x)$ since we can always write $F_S(\omega, \omega')$ in the form (2.6). They are merely a convenience that gives a simpler and more concise formula. In this sense, the complex numbers are convenient but not necessary. We have also derived the Feynman amplitude function $f_S(\omega) = N_S e^{iS(\omega)}$ (Feynman, 1949; Feynman and Hibbs, 1965) from deeper physical principles. Equation (2.8) justifies Feynman's prescription that the amplitude of an outcome x is the sum (integral) of the amplitudes of the configurations that result in x.

If $B \in \Sigma_X$, applying (2.4) and (2.9) gives the (X, S)-probability of B

$$P_{X,S}(B) = \int_B |f_{X,S}(x)|^2 \, \mu_X(dx) \tag{2.10}$$

It follows from (2.3) that $f_{X,S}$ is a unit vector in the Hilbert space $H_X = L^2(R(X), \Sigma_X, \mu_X)$ and this is where the Hilbert space comes from. Let us assume that $R(X) \subseteq \mathbb{R}$, which is the usual case for a measurement. Introducing the self-adjoint operator $\hat{X}$ on H_X defined by $\hat{X}g(x) = xg(x)$, we find that (2.10) becomes

$$P_{X,S}(B) = \|P^{\hat{X}}(B) f_{X,S}\|^2$$

where $P^{\hat{X}}$ is the spectral measure for $\hat{X}$. Hence, we obtain the self-adjoint operator representing the measurement X and have derived the usual probabilistic formula for its distribution.

A similar analysis applies to a superposition of two generalized actions. In this case (2.7) has the form

$$P_{X,S_1,S_2}(x) = \left| \int_{X^{-1}(x)\cap D_{S_1}\cap D_{S_2}} [ae^{iS_1(\omega)} + be^{iS_2(\omega)}]\, \mu_X^x(d\omega) \right|^2$$

The (S_1, S_2)-amplitude function then becomes

$$f_{S_1,S_2}(\omega) = \begin{cases} ae^{iS_1(\omega)} + be^{iS_2(\omega)} & \text{for} \quad \omega \in D_{S_1} \cap D_{S_2} \\ 0 & \text{for} \quad \omega \notin D_{S_1} \cap D_{S_2} \end{cases}$$

and the (X, S_1, S_2)-wave function is

$$f_{X,S_1,S_2}(x) = \int_{X^{-1}(x)} f_{S_1,S_2}(\omega)\, \mu_X^x(d\omega)$$

The probability density then becomes

$$P_{X,S_1,S_2}(x) = |f_{X,S_1,S_2}(x)|^2$$

Continuing the study of our probabilistic formulas, we define the (X, S)-*amplitude average at* x of a function $g: \Omega \to \mathbb{R}$ by

$$f_{X,S}(g)(x) = \int_{X^{-1}(x)} g(\omega) f_S(\omega)\, \mu_X^x(d\omega) \tag{2.11}$$

Then, applying (2.5), we obtain

$$E_{X,S}(g) = \operatorname{Re}\langle f_{X,S}(g), f_{X,S}\rangle \tag{2.12}$$

Equation (2.12) sometimes has the form $\langle Tf_{X,S}, f_{X,S}\rangle$, where T is a self-adjoint operator on H_X and hence g can be represented by a self-adjoint operator. We shall give examples of this in the next section. Finally, for $A \subseteq \Omega$, the (X, S)-pseudoprobability becomes

$$\hat{P}_{X,S}(A) = \operatorname{Re}\langle f_{X,S}(\chi_A), f_{X,S}\rangle \tag{2.13}$$

where, by (2.11),

$$f_{X,S}(\chi_A)(x) = \int_{X^{-1}(x)\cap A} f_S(\omega)\, \mu_X^x(d\omega) \tag{2.14}$$

3. TRADITIONAL QUANTUM MECHANICS

We now show that our formalism contains traditional nonrelativistic quantum mechanics. For simplicity, we consider a single spinless onedimensional particle, although this work easily generalizes to three dimensions. We take our sample space to be phase space

$$\Omega = \mathbb{R}^2 = \{(q, p) : q, p \in \mathbb{R}\}$$

The two most important measurements are the position and momentum, given by $Q(q, p) = q$, $P(q, p) = p$, respectively. However, as is frequently done in quantum mechanics, we shall investigate the Q-representation of the system. Then, instead of considering momentum as a measurement, we view $P: \Omega \to \mathbb{R}$ as a function on Ω.

Each Q-fiber, $Q^{-1}(q) = \{(q, p) : p \in \mathbb{R}\}$, can be identified with $\mathbb{R}$. Only certain measures on Q-fibers and certain generalized actions $S: \Omega \to \mathbb{R}$ correspond to traditional quantum mechanical states and these can be derived from natural postulates. We assume that μ_Q^q is absolutely continuous relative to Lebesgue measure on $\mathbb{R}$, and that μ_Q^q is independent of q. This is because sets of Lebesgue measure zero are too small to have any effect on the outcomes of position measurements and there is no *a priori* reason to distinguish between Q-fibers. It follows from the Radon–Nikodym theorem that there exists a nonnegative Lebesgue measurable function $\xi: \mathbb{R} \to \mathbb{R}$ such that

$$\mu_Q^q(dp) = (2\pi\hbar)^{-1/2}\xi(p)\, dp$$

With this measure on Q-fibers and Lebesgue measure on the range $R(Q) = \mathbb{R}$, Q is endowed with the structure of a measure in accordance with (M1) and (M2) of Section 2.

We now define the generalized action $S: \Omega \to \mathbb{R}$ by

$$S(q, p) = \frac{qp}{\hbar} + \eta(p) \tag{3.1}$$

Applying (2.8), we find that the (Q, S)-wave function becomes

$$f_{Q,S}(q) = (2\pi\hbar)^{-1/2} \int \xi(p) e^{i\eta(p)} e^{iqp/\hbar}\, dp$$

Defining $\phi(p) = \xi(p)e^{i\eta(p)}$ and $\psi(q) = \phi^{\vee}(p)$, where $^{\vee}$ denotes the inverse Fourier transform, we have

$$f_{Q,S}(q) = (2\pi\hbar)^{-1/2} \int \phi(p) e^{iqp/\hbar}\, dp = \psi(q) \tag{3.2}$$

It follows from (2.3) and (2.8) that ψ is a unit vector in the usual position Hilbert space $H_Q = L^2(\mathbb{R}, dq)$. Thus, ψ is the usual wave-function or state.

Applying (2.11), we obtain that the (Q, S)-amplitude average of P at q becomes

$$f_{Q,S}(P)(q) = (2\pi\hbar)^{-1}\int p\phi(p)e^{iqp/\hbar}\,dp$$

$$= -i\hbar\frac{d}{dq}(2\pi\hbar)^{-1/2}\int \phi(p)e^{ipq/\hbar}\,dp$$

$$= -i\hbar\frac{d\psi}{dq}(q)$$

More generally, if n is a positive integer, we obtain

$$f_{Q,S}(P^n)(q) = \left(-i\hbar\frac{d}{dq}\right)^n\psi(q) \tag{3.3}$$

Moreover, applying (2.12), we obtain

$$E_{Q,S}(P^n) = \int\left[\left(-i\hbar\frac{d}{dq}\right)^n\psi(q)\right]\psi^*(q)\,dq \tag{3.4}$$

which is the usual quantum expectation formula. We conclude from (3.3) or (3.4) that P^n corresponds to the operator $(-i\hbar\, d/dq)^n$.

Now let $V\colon \mathbb{R}\to\mathbb{R}$ and define $V(Q)\colon \Omega\to\mathbb{R}$ by $V(Q)(q,p) = V(q)$. For example, we may think of $V(Q)$ as a potential energy function. The (Q, S)-amplitude average of $V(Q)$ becomes

$$f_{Q,S}[V(Q)](q) = (2\pi\hbar)^{-1/2}\int V(q)\phi(p)e^{iqp/\hbar}\,dp$$

$$= V(q)\psi(q) \tag{3.5}$$

and (2.11) gives

$$E_{Q,S}[V(Q)] = \int V(q)\psi(q)\psi^*(q)\,dq \tag{3.6}$$

We conclude from (3.5) or (3.6) that $V(Q)$ corresponds to the operator which multiplies by $V(q)$. This together with our observation concerning P^n, gives a derivation of the Bohr correspondence principle.

We now consider probability distributions. Applying (2.1) for measurable $B\subseteq\mathbb{R}=R(Q)$, we have

$$P_{Q,S}(B) = \int_B |\psi(q)|^2\,dq$$

which is the usual distribution of Q. It is more interesting to compute the probability of $A = P^{-1}(B)$ for the momentum function P. We have from

(2.14) that

$$f_{Q,S}(\chi_A)(q) = (2\pi\hbar)^{-1/2} \int_B \phi(p) e^{ipq/\hbar}\, dp$$
$$= (\chi_B \phi)^{\vee}(q)$$

Hence, by (2.13) and the Plancherel formula we obtain

$$\hat{P}_{Q,S}[P^{-1}(B)] = \int (\chi_B \phi)^{\vee}(q) \psi^*(q)\, dq$$
$$= \int (\chi_B \phi)(p) \phi^*(p)\, dp$$
$$= \int_B |\hat{\psi}(p)|^2\, dp$$

Again, this is the usual momentum distribution. One can also derive Lüder's conditional probability formula and the Heisenberg uncertainty relations from this formalism (Gudder, 1988*b*, 1989).

Until now we have treated time as fixed. We now briefly consider dynamics. Let $\psi(q, t)$ be a smooth function. Our previous formulas hold with $\psi(q)$ replaced by $\psi(q, t)$ and μ_Q^q replaced by $\mu_{Q,t}^q$. We now derive Schrödinger's equation from Hamilton's equation of classical mechanics $dp/dt = -\partial H/\partial q$. Suppose that the energy measurement has the form

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

We now assume that Hamilton's equation holds in the amplitude average. Applying (2.11), we have

$$\frac{d}{dt} \int p f_S(q, p, t)\, \mu_{Q,t}^q(dp) = -\frac{\partial}{\partial q} \int H(q, p) f_S(q, p, t)\, \mu_{Q,t}^q(dp)$$

Hence,

$$\frac{d}{dt} \int p \hat{\psi}(p, t) e^{iqp/\hbar}\, dp = -\frac{\partial}{\partial q} \int H(q, p) \hat{\psi}(p, t) e^{iqp/\hbar}\, dp$$

Applying (3.3) and (3.5) gives

$$\frac{d}{dt}\left(-i\hbar \frac{\partial \psi}{\partial q}\right) = -\frac{\partial}{\partial q}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q^2} + V(q)\psi\right]$$

Interchanging the order of differentiation on the left side of this equation and integrating with respect to q gives Schrödinger's equation.

4. SPIN

We now show that spin can be described within the present framework. Since spin measurements have only a finite number of values, we shall not need the full generality of quantum probability as developed in Section 2. We first discuss a simplified version that is broad enough to include spin. Let (Ω, Σ, μ) be a measure space which we call a *measurable sample space*. Let $X: \Omega \to \mathbb{R}$ be measurable with only a finite number of values $R(X) = \{x_1, \ldots, x_n\}$ and let $v_1, \ldots, v_n$ be positive numbers. We then call $(X; v_1, \ldots, v_n)$ a *simple measurement*. For simplicity, we frequently denote a simple measurement by X. Note that in a natural way, X is a special case of a measurement as defined previously. In this case, the probability of the outcome x_i when X is executed becomes

$$P_{X,S}(x_i) \equiv P_{X,S}(\{x_i\}) = |f_{X,S}(x_i)|^2 v_i$$

We first consider the spin 1/2 case. Fix a direction corresponding to the z axis and assume that the spin j_z in the z direction is known (either 1/2 or $-1/2$). Let $\omega \in [0, \pi]$ denote a direction whose angle to the z axis is ω. By symmetry, the spin distribution should only depend upon ω. Define the measurable sample space $(\Omega_{1/2}, \Sigma_{1/2}, \mu_{1/2})$, where $\Omega_{1/2} = [0, \pi]$, $\Sigma_{1/2}$ is the Borel σ-algebra on $\Omega_{1/2}$, and $\mu_{1/2} = \frac{1}{2}\mu + \delta_0 + \delta_\pi$. The measure μ is Lebesgue measure and δ_0, δ_π are the Dirac point measures at $0, \pi$, respectively. The justification for this measure is that we have total ignorance on $(0, \pi)$ and precise knowledge at 0 and π.

For $\theta \in [0, \pi]$, define the function $\theta^{1/2}: \Omega_{1/2} \to \{1/2, -1/2\}$ by

$$\theta^{1/2}(\omega) = \begin{cases} -\frac{1}{2} & \text{for} \quad \omega \in [0, \theta] \\ \frac{1}{2} & \text{for} \quad \omega \in (\theta, \pi] \end{cases}$$

The function $\theta^{1/2}$ corresponds to a spin 1/2 measurement in the θ direction. Notice that $\theta^{1/2}$ is the simplest nontrivial function depending on θ that can be defined from $\Omega_{1/2}$ to $\{1/2, -1/2\}$. In this way, we obtain a collection of measurements $\{\theta^{1/2}: \theta \in [0, \pi]\}$ each being applied in a different direction. Observe that a sample point $\omega \in \Omega_{1/2}$ determines the spin in every direction simultaneously. To make $\theta^{1/2}$ a measurement, we must define positive weights $v_{1/2}, v_{-1/2}$ corresponding to the values $1/2, -1/2$. Pleading total ignorance, we let $v_{1/2} = v_{-1/2} = 1$.

For $j_z = 1/2$, we define the generalized action $S^{1/2}: (0, \pi) \to \mathbb{R}$ by $S^{1/2}(\omega) = \omega$. The $S^{1/2}$-amplitude function $f^{1/2} = f_{S^{1/2}}$ becomes

$$f^{1/2}(\omega) = \begin{cases} e^{i\omega} & \text{for} \quad \omega \in (0, \pi) \\ 0 & \text{for} \quad \omega \in \{0, \pi\} \end{cases}$$

The $(\theta^{1/2}, S^{1/2})$-wave function is given by

$$f_{\theta^{1/2}}^{1/2}\left(\frac{1}{2}\right) = \int_{(\theta^{1/2})^{-1}(1/2)} f^{1/2}\, d\mu_{1/2} = \frac{1}{2}\int_{\theta}^{\pi} e^{i\omega}\, d\omega = \frac{i}{2}(1 + e^{i\theta})$$

$$f_{\theta^{1/2}}^{1/2}\left(-\frac{1}{2}\right) = \int_{(\theta^{1/2})^{-1}(-1/2)} f^{1/2}\, d\mu_{1/2} = \frac{1}{2}\int_{0}^{\theta} e^{i\omega}\, d\omega = \frac{i}{2}(1 - e^{i\theta})$$

The probabilities are

$$P_{\theta^{1/2},S^{1/2}}\left(\frac{1}{2}\right) = \left|f_{\theta^{1/2}}^{1/2}\left(\frac{1}{2}\right)\right|^2 = \frac{1}{4}|1 + e^{i\theta}|^2 = \cos^2\frac{\theta}{2}$$

$$P_{\theta^{1/2},S^{1/2}}\left(-\frac{1}{2}\right) = \left|f_{\theta^{1/2}}^{1/2}\left(\frac{1}{2}\right)\right|^2 = \frac{1}{4}|1 - e^{i\theta}|^2 = \sin^2\frac{\theta}{2}$$

Of course, this is the usual probability distribution for the spin in the θ direction when $j_z = 1/2$.

For $j_z = -1/2$, we define the generalized action $S^{-1/2}$: $[0, \pi] \to \mathbb{R}$ by

$$S^{-1/2}(\omega) = \begin{cases} \omega & \text{for} \quad \omega \in (0, \pi) \\ -\pi/2 & \text{for} \quad \omega \in \{0, \pi\} \end{cases}$$

The $S^{-1/2}$-amplitude function $f^{-1/2} = f_{S^{-1/2}}$ becomes

$$f^{-1/2}(\omega) = \begin{cases} e^{i\omega} & \text{for} \quad \omega \in (0, \pi) \\ -i & \text{for} \quad \omega \in \{0, \pi\} \end{cases}$$

The $(\theta^{1/2}, S^{-1/2})$-wave function is given by

$$f_{\theta^{1/2}}^{-1/2}\left(\frac{1}{2}\right) = \int_{(\theta^{1/2})^{-1}(1/2)} f^{-1/2}\, d\mu_{1/2} = -\frac{i}{2}(1 - e^{i\theta})$$

$$f_{\theta^{1/2}}^{-1/2}\left(-\frac{1}{2}\right) = \int_{(\theta^{1/2})^{-1}(-1/2)} f^{-1/2}\, d\mu_{1/2} = -\frac{i}{2}(1 + e^{i\theta})$$

We then have

$$P_{\theta^{1/2},S^{-1/2}}\left(\frac{1}{2}\right) = \sin^2\frac{\theta}{2}$$

$$P_{\theta^{1/2},S^{-1/2}}\left(-\frac{1}{2}\right) = \cos^2\frac{\theta}{2}$$

which is again the usual distribution for spin in the θ direction when $j_z = -1/2$.

We next consider the spin-1 case. Roughly speaking, a spin-1 measurement can be considered as a sum of two spin 1/2 measurements. Equivalently, a spin-1 particle is viewed as a system composed of two spin 1/2 particles. We take the spin-1 measurable sample space $(\Omega_1, \Sigma_1, \mu_1)$ to be the Cartesian product

$$(\Omega_{1/2} \times \Omega_{1/2}, \Sigma_{1/2} \times \Sigma_{1/2}, \mu_{1/2} \times \mu_{1/2})$$

of two spin-1/2 sample spaces and define $\theta^1: \Omega_1 \to \{1, 0, -1\}$ by $\theta^1(\omega_1, \omega_2) = \theta^{1/2}(\omega_1) + \theta^{1/2}(\omega_2)$, where $\theta^{1/2}$ is the spin-1/2 measurement defined previously. On the range $R(\theta^1) = \{1, 0, -1\}$ we define the weights $\nu_1 = \nu_{-1} = 1$, and $\nu_0 = 1/2$. We place the weight 1/2 on the value 0, since 0 can occur in two different ways and this gives each of the values an equivalent weight.

For $j_z = 1$, we define the generalized action $S^1: (0, \pi) \times (0, \pi) \to \mathbb{R}$ by

$$S^1(\omega_1, \omega_2) = S^{1/2}(\omega_1) + S^{1/2}(\omega_2) = \omega_1 + \omega_2$$

The S^1-amplitude function $f^1 = f_{S^1}$ becomes

$$f^1(\omega_1, \omega_2) = f^{1/2}(\omega_1) f^{1/2}(\omega_2)$$

where $f^{1/2}$ was given previously. The (θ^1, S^1)-wave function is

$$\begin{aligned}
f^1_{\theta^1}(1) &= \int_{(\theta^1)^{-1}(1)} f^1 \, d\mu_1 \\
&= \frac{1}{4} \int_\theta^\pi e^{i\omega_1} \, d\omega_1 \int_\theta^\pi e^{i\omega_2} \, d\omega_2 = -\frac{1}{4}(1 + e^{i\theta})^2 \\
f^1_{\theta^1}(-1) &= \int_{(\theta^1)^{-1}(-1)} f^1 \, d\mu_1 \\
&= \frac{1}{4} \int_0^\theta e^{i\omega_1} \, d\omega_1 \int_0^\theta e^{i\omega_2} \, d\omega_2 = -\frac{1}{4}(1 - e^{i\theta})^2 \\
f^1_{\theta^1}(0) &= \int_{(\theta^1)^{-1}(0)} f^1 \, d\mu_1 \\
&= \frac{2}{4} \int_\theta^\pi e^{i\omega_1} \, d\omega_1 \int_0^\theta e^{i\omega_2} \, d\omega_2 = -\frac{1}{2}(1 + e^{i\theta})(1 - e^{i\theta})
\end{aligned}$$

The probabilities become

$$P_{\theta^1, S^1}(1) = |f^1_{\theta^1}(1)|^2 = \cos^4 \frac{\theta}{2}$$

$$P_{\theta^1, S^1}(-1) = |f^1_{\theta^1}(-1)|^2 = \sin^4 \frac{\theta}{2}$$

$$P_{\theta^1, S^1}(0) = \frac{1}{2} |f^1_{\theta^1}(0)|^2 = \frac{1}{2} \sin^2 \theta$$

which is the usual distribution.

For $j_z = -1$, we define the generalized action S^{-1}: $[0, \pi] \times [0, \pi] \to \mathbb{R}$ by

$$S^{-1}(\omega_1, \omega_2) = S^{-1/2}(\omega_1) + S^{-1/2}(\omega_2)$$

As before, the S^{-1}-amplitude function $f^{-1} = f_{S^{-1}}$ becomes $f^{-1}(\omega_1, \omega_2) = f^{-1/2}(\omega_1) f^{-1/2}(\omega_2)$. By a calculation similar to the previous one we obtain

$$P_{\theta 1, S^{-1}}(1) = \sin^4 \frac{\theta}{2}$$

$$P_{\theta 1, S^{-1}}(-1) = \cos^4 \frac{\theta}{2}$$

$$P_{\theta 1, S^{-1}}(0) = \frac{1}{2} \sin^2 \theta$$

which is the usual distribution.

In the $j_z = 0$ case, we define the generalized action S^0: $(0, \pi) \times [0, \pi] \to \mathbb{R}$ by

$$S^0(\omega_1, \omega_2) = S^{1/2}(\omega_1) + S^{-1/2}(\omega_2)$$

We now need the normalization constant $N_{S^0} = \sqrt{2}$ and the S^0-amplitude function $f^0 = f_{S^0}$ becomes

$$f^0(\omega_1, \omega_2) = \sqrt{2} f^{1/2}(\omega_1) f^{-1/2}(\omega_2)$$

We then obtain

$$P_{\theta 1, S^0}(1) = P_{\theta 1, S^0}(-1) = \frac{1}{2} \sin^2 \theta$$

$$P_{\theta 1, S^0}(0) = \cos^2 \theta$$

which is again the usual distribution.

We can continue this process in the natural way and obtain spin measurements of any order. It can be shown that these measurements reproduce the usual quantum distributions (Gudder, 1993).

We now show that this framework gives a realistic solution to the EPR problem. In this problem, two spin-1/2 particles are created in the singlet state which gives correlated measurement values even at large separation. We do this by constructing a model for a combined system of two spin-1/2 particles. Let $(\Omega_1, \Sigma_1, \mu_1)$ be the spin-1 sample space defined previously. We now consider this as the sample space describing two spin-1/2 particles. For $\theta_1, \theta_2 \in [0, \pi]$, we define the measurements $\theta_{11}^{1/2}, \theta_{22}^{1/2}$ by

$$\theta_{11}^{1/2}(\omega_1, \omega_2) = \theta_1^{1/2}(\omega_1)$$

$$\theta_{22}^{1/2}(\omega_1, \omega_2) = \theta_2^{1/2}(\omega_2)$$

where $\theta_1^{1/2}$, $\theta_2^{1/2}$ are the spin-1/2 measurements defined previously. Then $\theta_{11}^{1/2}$ gives a spin-1/2 measurement of particle 1 in the θ_1 direction and $\theta_{22}^{1/2}$ gives a spin-1/2 measurement of particle 2 in the θ_2 direction.

The singlet state is given by the generalized action $S: D_S \to \mathbb{R}$ with domain

$$D_S = ((0, \pi) \times \{0, \pi\}) \cup (\{0, \pi\} \times (0, \pi))$$

and defined by

$$S(\omega_1, \omega_2) = \begin{cases} \omega_1 + \pi/2 & \text{for} \quad (\omega_1, \omega_2) \in (0, \pi) \times \{0, \pi\} \\ \omega_2 - \pi/2 & \text{for} \quad (\omega_1, \omega_2) \in \{0, \pi\} \times (0, \pi) \end{cases}$$

It is easy to show that, the normalization constant $N_S = 1/\sqrt{2}$, the S-amplitude function becomes

$$f_S(\omega_1, \omega_2) = \frac{1}{\sqrt{2}} [f^{-1/2}(\omega_1) f^{1/2}(\omega_2) - f^{1/2}(\omega_1) f^{-1/2}(\omega_2)]$$

A simple calculation gives the $(\theta_{11}^{1/2}, S)$-wave function

$$f_{\theta_{11}^{1/2}, S}\left(\frac{1}{2}\right) = \int_{(\theta_{11}^{1/2})^{-1}(1/2)} f_S \, d\mu_1 = \frac{1}{\sqrt{2}} e^{i\theta_1}$$

$$f_{\theta_{11}^{1/2}, S}\left(-\frac{1}{2}\right) = \int_{(\theta_{11}^{1/2})^{-1}(-1/2)} f_S \, d\mu_1 = -\frac{1}{\sqrt{2}} e^{i\theta_1}$$

Hence,

$$P_{\theta_{11}^{1/2}, S}\left(\frac{1}{2}\right) = P_{\theta_{11}^{1/2}, S}\left(-\frac{1}{2}\right) = \frac{1}{2}$$

This is the usual distribution for the measurement $\theta_{11}^{1/2}$. A similar result holds for $\theta_{22}^{1/2}$.

To obtain the spin correlations, we define the four sets

$$A^{\pm\pm} = (\theta_{11}^{1/2})^{-1}\left(\pm\frac{1}{2}\right) \cap (\theta_{22}^{1/2})^{-1}\left(\pm\frac{1}{2}\right)$$

It is natural to define the *S-amplitudes* of these sets as

$$f_S(A^{\pm\pm}) = \int_{A^{\pm\pm}} f_S \, d\mu_1$$

A straightforward computation then gives

$$f_S(A^{++}) = -f(A^{--}) = 2^{-3/2}(e^{i\theta_1} - e^{i\theta_2})$$

$$f_S(A^{+-}) = -f(A^{-+}) = 2^{-3/2}(e^{i\theta_1} + e^{i\theta_2})$$

It is again natural to define the *S-joint probability distributions* of $\theta_{11}^{1/2}$ and $\theta_{22}^{1/2}$ by

$$P_{\theta_{11}^{1/2},\theta_{22}^{1/2},S}\left(\pm\frac{1}{2},\pm\frac{1}{2}\right)=|f_S(A^{\pm\pm})|^2$$

We then obtain

$$P_{\theta_{11}^{1/2},\theta_{22}^{1/2},S}\left(\frac{1}{2},\frac{1}{2}\right)=P_{\theta_{11}^{1/2},\theta_{22}^{1/2},S}\left(-\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{2}\sin^2\frac{\theta_2-\theta_1}{2}$$

$$P_{\theta_{11}^{1/2},\theta_{22}^{1/2},S}\left(\frac{1}{2},-\frac{1}{2}\right)=P_{\theta_{11}^{1/2},\theta_{22}^{1/2},S}\left(-\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\cos^2\frac{\theta_2-\theta_1}{2}$$

This is the usual quantum joint distribution, which shows that there is correlation between the two measurements.

We have thus constructed a realistic model for the EPR problem that gives the same predictions as traditional quantum mechanics. The reason this model is not contradicted by Bell's theorem is that Bell's inequalities were derived assuming classical probability theory and we have employed quantum probability theory (Gudder, 1994).

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